Diagonalization of Linear Operators on Inner Product Spaces

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Let $T : V \rightarrow V$ be a linear operator on an inner product space $V$.

What conditions guarantee that $V$ has an orthonormal basis of eigenvectors of $T$?
When $T$ has an eigenvector, so does $T^*$

Lemma
Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional inner product space $V$.
If $T$ has an eigenvector, then so does $T^*$.

Proof
Suppose that $\vec{v}$ is an eigenvector of $T$ with corresponding eigenvalue $\lambda$. Then for any $\vec{x} \in V$,

$$0 = \langle \vec{0}, \vec{x} \rangle = \langle (T - \lambda I)(\vec{v}), \vec{x} \rangle = \langle \vec{v}, (T - \lambda I)^*(\vec{x}) \rangle = \langle \vec{v}, (T^* - \bar{\lambda} I)(\vec{x}) \rangle,$$

and hence $\vec{v}$ is orthogonal to the range of $T^* - \bar{\lambda} I$.

So $T^* - \bar{\lambda} I$ is not onto and hence not one-to-one.
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Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional inner product space $V$.
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**Proof**
So $T^* - \lambda I$ is not onto and hence not one-to-one.
Thus $T^* - \lambda I$ has a nonzero null space, and any nonzero vector in this null space is an eigenvector of $T^*$ with corresponding eigenvalue $\lambda$. 
Triangularizability

Theorem (Schur)

Let $T$ be a linear operator on a finite-dimensional inner product space $V$. Suppose that the characteristic polynomial of $T$ splits. Then there exists an orthonormal basis $\beta$ for $V$ such that the matrix $[T]_{\beta}$ is upper triangular.
Proof of Schur’s Theorem

Proceed by induction on \( n = \dim(V) \), noting that the case \( n = 1 \) is immediate.
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Proof of Schur’s Theorem

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Suppose the result is true for linear operators on $(n - 1)$-dimensional inner product spaces whose characteristic polynomials split.

Since $T$ has an eigenvector, so does $T^*$ by the prior lemma. Write $T^*(\vec{z}) = \lambda \vec{z}$ and let $W = \text{Span}(\{\vec{z}\})$. 

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Since \( T \) has an eigenvector, so does \( T^* \) by the prior lemma. Write \( T^* (\vec{z}) = \lambda \vec{z} \) and let \( W = \text{Span}(\{\vec{z}\}) \).

Observe that \( W^\perp \) is \( T \)-invariant. It follows that the characteristic polynomial of \( T_{W^\perp} \) splits since it divides the characteristic polynomial of \( T \). Further, \( \dim(W^\perp) = n - 1 \).
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Now apply the result to \( T_{W^\perp} \) to obtain an orthonormal basis \( \gamma \) of \( W^\perp \) such that \( [T_{W^\perp}]_{\gamma} \) is upper triangular.
Proof of Schur’s Theorem

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Observe that $W^\perp$ is $T$-invariant. It follows that the characteristic polynomial of $T_{W^\perp}$ splits since it divides the characteristic polynomial of $T$. Further, $\dim(W^\perp) = n - 1$.

Now apply the result to $T_{W^\perp}$ to obtain an orthonormal basis $\gamma$ of $W^\perp$ such that $[T_{W^\perp}]_\gamma$ is upper triangular.

We find that $\beta = \gamma \cup \{\vec{z}\}$ is an orthonormal basis for $V$ such that $[T]_\beta$ is upper triangular.
A Necessary Condition for Existence of an Orthonormal Basis of Eigenvectors of $T$

Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional inner product space $V$.

Suppose that $V$ has an orthonormal basis $\beta$ of eigenvectors of $T$.

Then $[T]_\beta$ is diagonal, and we have that $[T^*]_\beta = [T]^*_\beta$ is also diagonal.

Since diagonal matrices commute, we infer that $T$ and $T^*$ commute.
A Necessary Condition for Existence of an Orthonormal Basis of Eigenvectors of $T$

Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional inner product space $V$.

**Proposition**

If $V$ possesses an orthonormal basis of eigenvectors of $T$, then $T T^* = T^* T$.

**Definition**

We say that a linear operator $T$ on an inner product space is normal when $T T^* = T^* T$. 
Is normality of a linear operator $T$ on an inner product space $V$ sufficient for existence of an orthonormal basis of $V$ consisting of eigenvectors of $T$?
Example

Consider the real inner product space $\mathbb{R}^2$ with dot product and standard orthonormal basis $\beta$.

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation by $\theta$ where $0 < \theta < \pi$.

Observe that $[T]_\beta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.

Writing $A = [T]_\beta$, verify that $AA^* = I = A^* A$ and that $A$ has no eigenvector.
Properties of Normal Operators

Theorem
Let $V$ be an inner product space and let $T$ be a normal operator on $V$. Then the following statements are true:

(a) $\|T(\vec{x})\| = \|T^*(\vec{x})\|$ for all $\vec{x} \in V$

(b) $T - cI$ is normal for every $c \in F$

(c) If $\vec{x}$ is an eigenvector of $T$, then $\vec{x}$ is also an eigenvector of $T^*$. In fact, if $T(\vec{x}) = \lambda \vec{x}$, then $T^*(\vec{x}) = \bar{\lambda} \vec{x}$.

(d) If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of $T$ with corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$, then $\vec{x}_1$ and $\vec{x}_2$ are orthogonal.
Properties of Normal Operators

**Theorem**

Let $V$ be an inner product space and let $T$ be a normal operator on $V$. Then the following statements are true:

(a) $\| T(\vec{x}) \| = \| T^*(\vec{x}) \|$ for all $\vec{x} \in V$

Recall that

$$\langle T(\vec{u}), \vec{v} \rangle = \langle \vec{u}, T^*(\vec{v}) \rangle$$

and

$$\langle \vec{u}, T(\vec{v}) \rangle = \langle T^*(\vec{u}), \vec{v} \rangle.$$

For each $\vec{x} \in V$ we then have

$$\| T(\vec{x}) \|^2 = \langle T(\vec{x}), T(\vec{x}) \rangle = \langle T^* T(\vec{x}), \vec{x} \rangle$$

$$= \langle T T^*(\vec{x}), \vec{x} \rangle = \langle T^*(\vec{x}), T^*(\vec{x}) \rangle$$

$$= \| T^*(\vec{x}) \|^2.$$
Properties of Normal Operators

Theorem
Let $V$ be an inner product space and let $T$ be a normal operator on $V$. Then the following statements are true:

(c) If $\vec{x}$ is an eigenvector of $T$, then $\vec{x}$ is also an eigenvector of $T^*$. In fact, if $T(\vec{x}) = \lambda \vec{x}$, then $T^*(\vec{x}) = \overline{\lambda} \vec{x}$.

Suppose $T(\vec{x}) = \lambda \vec{x}$ for some nonzero $\vec{x} \in V$. Let $U = T - \lambda I$. Then $U(\vec{x}) = \vec{0}$ and $U$ is normal by part (b).

Using part (a), we see that

$$0 = \|U(\vec{x})\| = \|U^*(\vec{x})\| = \|(T^* - \overline{\lambda} I)(\vec{x})\| = \|T^*(\vec{x}) - \overline{\lambda} \vec{x}\|.$$ 

Thus $T^*(\vec{x}) = \overline{\lambda} \vec{x}$ and we have that $\vec{x}$ is an eigenvector of $T^*$.
Properties of Normal Operators

**Theorem**

Let $V$ be an inner product space and let $T$ be a normal operator on $V$. Then the following statements are true:

(d) If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of $T$ with corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$, then $\vec{x}_1$ and $\vec{x}_2$ are orthogonal.

Observe that

$$
\lambda_1 \langle \vec{x}_1, \vec{x}_2 \rangle = \langle \lambda_1 \vec{x}_1, \vec{x}_2 \rangle = \langle T(\vec{x}_1), \vec{x}_2 \rangle = \langle \vec{x}_1, T^*(\vec{x}_2) \rangle = \langle \vec{x}_1, \lambda_2 \vec{x}_2 \rangle = \lambda_2 \langle \vec{x}_1, \vec{x}_2 \rangle.
$$

Since $\lambda_1 \neq \lambda_2$, we must have $\langle \vec{x}_1, \vec{x}_2 \rangle = 0.$
Normality Suffices for Complex Inner Product Spaces

Theorem
Let $T$ be a linear operator on a finite-dimensional complex inner product space $V$. Then $T$ is normal if and only if there exists an orthonormal basis for $V$ consisting of eigenvectors of $T$.

Proof Idea
Observe that necessity has already been justified.

Since the inner product space is complex, we know from the Fundamental Theorem of Algebra that the characteristic polynomial of $T$ splits. We may then apply Schur’s Theorem to obtain an orthonormal basis $\beta$ for $V$ such that $[T]_{\beta} = A$ is upper triangular.

Now proceed by induction on $n = \dim(V)$. 
What Condition Suffices for Real Inner Product Spaces?

We claim that the condition $T = T^*$ is sufficient for the existence of an orthonormal basis of eigenvectors of $T$ when $V$ is a real inner product space.

**Definition**

Let $T$ be a linear operator on an inner product space $V$. We say that $T$ is **self-adjoint** (or **Hermitian**) when $T = T^*$.

Observe that self-adjointness is a stronger condition than normality.
Example

Determine whether the linear operator is normal, self-adjoint, or neither.

Let $V = \mathbb{C}^2$ with standard inner product and define $T : V \to V$ by $T(a, b) = (2a + ib, a + 2b)$. 
Example

Determine whether the linear operator is normal, self-adjoint, or neither.

Let $V = \mathbb{C}^2$ with standard inner product and define $T : V \rightarrow V$ by $T(a, b) = (2a + i b, -i a + 2b)$. 
Example

Determine whether the linear operator is normal, self-adjoint, or neither.

Let $V = P_2(\mathbb{R})$ with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt$ and define $T : V \to V$ by $T(f) = f'$. 

Example

Determine whether the linear operator is normal, self-adjoint, or neither.

Let $V = P_2(\mathbb{R})$ with inner product $\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt$ and define $T : V \to V$ by $T(f) = f'$.

Take an orthonormal basis for $P_2(\mathbb{R})$:

For $\beta = \{1, \sqrt{3}(2t - 1), \sqrt{5}(6t^2 - 6t + 1)\}$, we have

$$[T]_\beta = \begin{bmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{bmatrix}.$$
Real Eigenvalues

Lemma
Let $T$ be a self-adjoint operator on a finite-dimensional inner product space $V$. Then every eigenvalue of $T$ is real.

Proof
Suppose that $T(\vec{x}) = \lambda \vec{x}$ for some $\vec{x} \neq \vec{0}$.

Since self-adjointness implies normality, we may recall a property of normal operators to obtain

$$\lambda \vec{x} = T(\vec{x}) = T^*(\vec{x}) = \overline{\lambda} \vec{x}.$$  

It follows that $\lambda = \overline{\lambda}$, so that $\lambda$ is real.

An immediate consequence of the lemma is that for self-adjoint operators on finite-dimensional inner product spaces, the characteristic polynomial splits.
Self-Adjointness Suffices for Real Inner Product Spaces

**Theorem**
Let $T$ be a linear operator on a finite-dimensional real inner product space $V$. Then $T$ is self-adjoint if and only if there exists an orthonormal basis $\beta$ for $V$ consisting of eigenvectors of $T$.

**Proof**
The argument for necessity is immediate.

Suppose that $T$ is self-adjoint. By the lemma, the characteristic polynomial of $T$ splits. Hence we may invoke Schur’s Theorem to obtain an orthonormal basis $\beta$ for $V$ such that $A = [T]_\beta$ is upper triangular.

Since $A = A^*$, we must have that $A$ is diagonal. Thus $\beta$ consists of eigenvectors of $T$. 