Bipartite and Complete Graphs
Definition of Graph

Definition
A graph $G = (V, E)$ is a structure consisting of a finite set $V$ of vertices (also known as nodes) and a finite set $E$ of edges such that each edge $e$ is associated with a pair of vertices $v$ and $w$.

We write $e = \{v, w\}$ or $\{w, v\}$ and say that:

1. $e$ is an edge between $v$ and $w$,
2. $e$ is incident on both $v$ and $w$, and
3. $e$ joins the vertices $v$ and $w$.

In this case both $v$ and $w$ are adjacent vertices and they are incident on $e$. 
Subgraphs and Induced Subgraphs

Definition
Let $G = (V, E)$ be a graph. A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V'$ is a subset of $V$ and $E'$ is a subset of $E$.

If $W$ is any subset of $V$, the subgraph of $G$ induced by $W$ is the graph $H = (W, F)$ where $f$ is an edge in $F$ if $f = \{u, v\}$ where $f$ is in $E$ and both $u$ and $v$ are in $W$. 
Definition
A graph is said to be **connected** if there is a path between every pair of vertices in it.
Connected Components of Graphs

**Proposition**
Let $G = (V, E)$ be a graph and let $\sim$ be the relation on $V$ defined by $v \sim w$ if and only if $v$ and $w$ are connected by a path. Then $\sim$ is an equivalence relation on $V$.

**Definition**
Let $\tilde{V}$ be an equivalence class of the relation $\sim$ on $V$. The subgraph induced by $\tilde{V}$ is called a **connected component** of the graph.

**Note**
A connected component of a graph is a maximal connected subgraph.
Complete Graphs

Definition
A simple graph with $n$ vertices is said to be complete if there is an edge between every pair of vertices.

The complete graph on $n$ vertices is denoted by $K_n$.

Proposition
The number of edges in $K_n$ is $\frac{n(n-1)}{2}$.
Bipartite Graphs

Definition
A **bipartite graph** is a graph in which the vertices can be partitioned into two disjoint sets $V$ and $W$ such that each edge is an edge between a vertex in $V$ and a vertex in $W$. 
Example

This graph is bipartite.
Subgraphs of Bipartite Graphs

Proposition

A subgraph of a bipartite graph is bipartite.
Example

This graph is not bipartite.
Example

Is this graph bipartite?
Some Families of Bipartite Graphs

Proposition
A path $P_n$ of length $n - 1$ is bipartite.

Proposition
A cycle $C_n$ of length $n$ is bipartite if and only if $n$ is even.
Complete Bipartite Graphs

Definition
A complete bipartite graph is a simple graph in which the vertices can be partitioned into two disjoint sets $V$ and $W$ such that each vertex in $V$ is adjacent to each vertex in $W$.

Notation
If $|V| = m$ and $|W| = n$, the complete bipartite graph is denoted by $K_{m,n}$.

Proposition
The number of edges in $K_{m,n}$ is $mn$. 
Cycle Characterization of Connected Bipartite Graphs

Theorem
A connected graph is bipartite if and only if it contains no cycle of odd length.

Proof Idea
(⇒) by contrapositive

(⇐) Choose a vertex \( v \) and partition the vertex set by parity of length of shortest path from \( v \).
The Number of Edges of a Bipartite Graph

Let $G$ be a simple bipartite graph with $t$ vertices.

What is the largest number of edges that $G$ can have?
Acknowledgements

Statements of definitions follow the notation and wording of Balakrishnan’s *Introductory Discrete Mathematics*. 